Episode 20

Energy and Momentum for Rigid Bodies

ENGN0040: Dynamics and Vibrations

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Topics for todays class

Conservation Laws for Systems of Rigid Bodies

1. Work-Power-Energy relations for rigid bodies
2. Angular Impulse – Angular Momentum relations for rigid bodies

Examples of 3D motion of rigid bodies (optional)

1. Steady precession of a spinning wheel
2. Boomerangs
3. Spinning wheel on a turntable

6.6 Work-Energy relations for systems of rigid bodies

Rigid bodies = distributions of particles

$\Rightarrow$ we can use energy equations for systems of particles

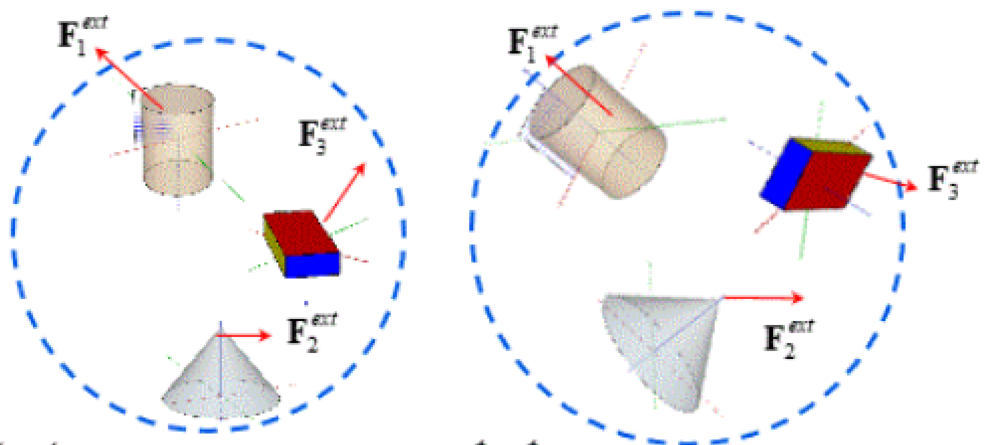
$$P^{ext} = \frac{d}{dt}(T+U)$$

$$\Delta W^{ext} = (T_1+U_1) - (T_0+U_0)$$

Use rigid body formulas for T

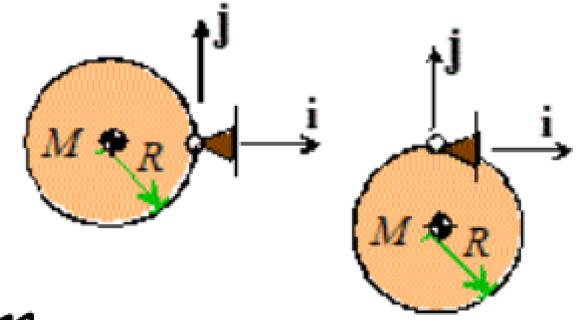
External Power $P^{ext}(t)$

External work $\Delta W^{ext} = \int_{t0}^{t1} P(t)\,dt$



$t=t_0$	$t=t_1$
Total KE T_0^{TOT}	*Total KE* T_1^{TOT}
Total KE U_0^{TOT}	*Total KE* U_1^{TOT}

Example 6.6.1: The disk swings freely about the pivot at O. It is released from rest with COM level with O. Find its angular speed when the COM is vertically below O.



System = disk + earth

$\Rightarrow \Delta W^{ext} = 0$; Conservative system

$\Rightarrow T_1 + U_1 = T_0 + U_0$

$T_0 = 0 \qquad U_0 = 0$ (datum for PE @ pivot)

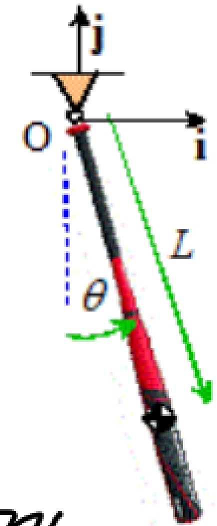
$T_1 = \frac{1}{2} I_{Ozz}\,\omega_z^2 \qquad U_1 = -MgR$

Parallel Axis $\Rightarrow I_{Ozz} = \frac{1}{2}MR^2 + MR^2 = \frac{3MR^2}{2}$

$T_1 + U_1 = T_0 + U_0 \Rightarrow \frac{3}{4} MR^2 \omega_z^2 - MgR = 0$

$$\Rightarrow \boxed{\omega_z = 2\sqrt{\frac{g}{3R}}}$$

Example 6.6.2: ASTM standard test to determine the 'sweet spot' on a bat:
(1) Measure the period τ of oscillations of the bat suspended from the handle
(2) Calculate the distance of the 'sweet spot' from the handle using $d = \tau^2 g / (4\pi^2)$
Why does this work?



Recall $d = I_{Ozz} / (mL)$ (ex 6.5.10)

Find formula for τ by analyzing vibrations

Energy method

$$T = \frac{1}{2} I_{Ozz}\,\omega_z^2 = \frac{1}{2} I_{Ozz}\left(\frac{d\theta}{dt}\right)^2 \qquad U = -mgL\cos\theta$$

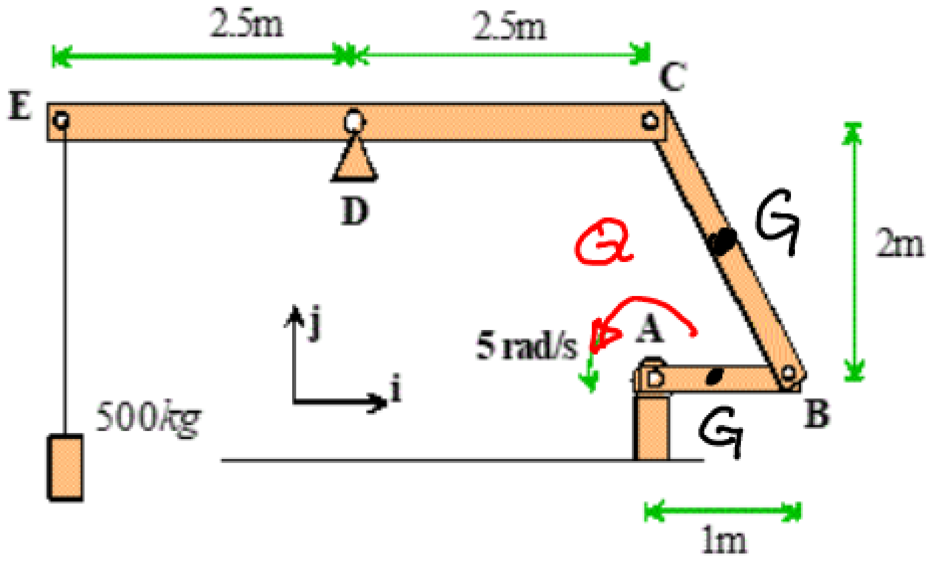
$$\frac{d}{dt}(T+U) = I_{Ozz}\cancel{\frac{d\theta}{dt}}\,\frac{d^2\theta}{dt^2} + mgL\sin\theta\,\cancel{\frac{d\theta}{dt}} = 0$$

$$\sin\theta \approx \theta \Rightarrow \underbrace{\frac{I_{Ozz}}{mgL}}_{\frac{1}{\omega_n^2}}\frac{d^2\theta}{dt^2} + \theta = 0 \qquad \Rightarrow \frac{1}{\omega_n^2} = \frac{I_{Ozz}}{mgL}$$

Recall $\tau = \dfrac{2\pi}{\omega_n} \Rightarrow \boxed{\dfrac{\tau^2 g}{4\pi^2} = \dfrac{g}{\omega_n^2} = \dfrac{I_{Ozz}}{mL} = d}$

ASTM WORKS!

Example 6.6.3: The bars have mass per unit length 50 kg/m. The mass attached to E has mass 500kg. Find the motor torque at A at the instant shown.



System = earth + jack

$\Rightarrow P^{ext} = Q\omega = \frac{d}{dt}(T+U)$

Preliminary Calcs

(1) Masses: $m_{AB} = 50 \quad m_{BC} = 50\sqrt{5} \quad m_{CE} = 50 \times 5$

(2) Motion (see ex 6.2.2)

$\omega_{BC} = 0 \quad \alpha_{BC} = -15/2 \quad \omega_{CE} = 2 \quad \alpha_{CE} = 3 \quad \underline{v}_E = -5\underline{j}$

$\underline{a}_E = 10\underline{i} - 7.5\underline{j}$

We also need $\underline{v}_G, \underline{a}_G$ for AB, BC

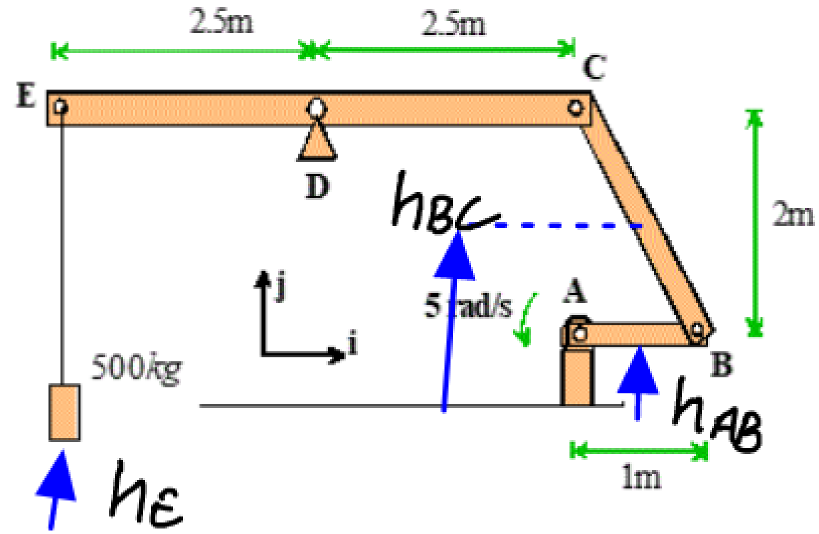
AB: Circular motion $\Rightarrow \underline{v}_{GAB} = 5/2\,\underline{j} \quad \underline{a}_{GAB} = -25/2\,\underline{i}$

BC: $\underline{v}_{GBC} - \underline{v}_B = \omega_{BC}\underline{k} \times (\underline{r}_{GBC} - \underline{r}_B) = \underline{0}$

$\Rightarrow \underline{v}_{GBC} = \underline{v}_B = 5\underline{j}$

$\underline{a}_{GBC} - \underline{a}_B = \alpha_{BC}\underline{k} \times (\underline{r}_{GBC} - \underline{r}_B) - 0 = -15/2\,\underline{k} \times (-1/2\,\underline{i} + \underline{j})$

$\Rightarrow \underline{a}_{GBC} = (-25 + 15/2)\underline{i} + 15/4\,\underline{j} = -35/2\,\underline{i} + 15/4\,\underline{j}$



$$U = m_{AB} g h_{AB} + m_{BC} g h_{BC} + m_E g h_E$$

$$\Rightarrow \frac{dU}{dt} = m_{AB} g V_{GABy} + m_{BC} g V_{GBCy} + m_E g V_{Ey}$$

$$\Rightarrow dU/dt = -17.8 \text{ kW}$$

$$T = \frac{1}{2} I_{ozz}^{AB} \omega_{AB}^2 + \frac{1}{2} m_{BC} \underline{V}_{BC} \cdot \underline{V}_{BC} + \frac{1}{2}(m_{BC} L_{BC}^2/12)\, \omega_{BC}^2 + \frac{1}{2}(m_{CE} L_{CE}^2/12)\, \omega_{CE}^2 + \frac{1}{2} m_E \underline{V}_E \cdot \underline{V}_E$$

$$\Rightarrow \frac{dT}{dt} = I_{ozz}^{AB} \omega_{AB} \alpha_{AB} + m_{BC} \underline{V}_{BC} \cdot \underline{a}_{BC} + (m_{BC} L_{BC}^2/12)\, \omega_{BC} \alpha_{BC} + (m_{CE} L_{CE}^2/12)\, \omega_{CE} \alpha_{CE} + m_E \underline{V}_E \cdot \underline{a}_E$$

$$\Rightarrow (dT/dt) = 23.9 \text{ kW} \Rightarrow \boxed{Q = \frac{1}{\omega_{AB}}\left(\frac{dU}{dt} + \frac{dT}{dt}\right) = 1.2 \text{ kNm}}$$

6.7 Angular Impulse - Momentum for systems of rigid bodies

We can use equations for systems of particles

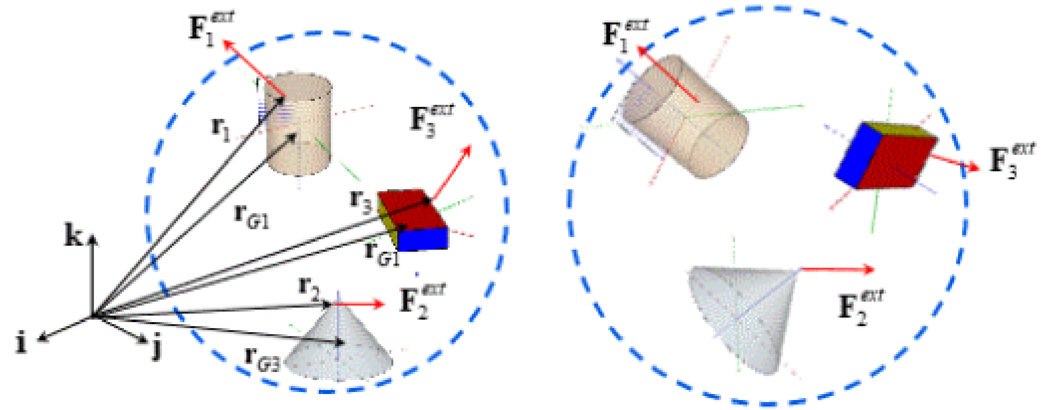
$$\sum \underline{r} \times \underline{F}^{ext} + \sum \underline{Q} = \frac{d\underline{h}^{TOT}}{dt}$$

$$\underline{A}^{ext} = \underline{h}_1^{TOT} - \underline{h}_0^{TOT}$$

Use rigid body formulas for $\underline{h}$

External Moment $\sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext} + \sum_i \mathbf{Q}$

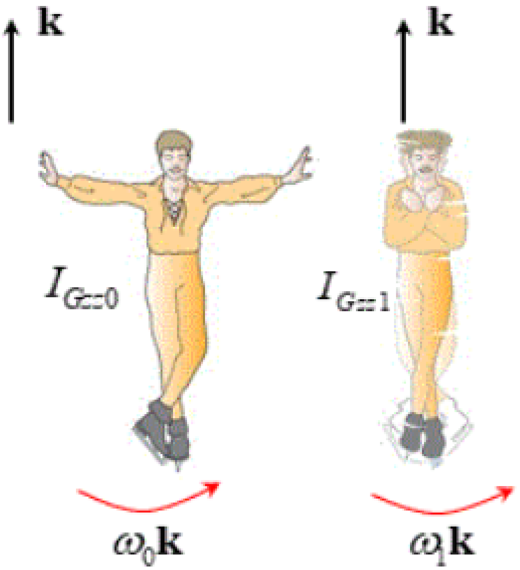
External Angular Impulse $\mathbf{A}^{ext} = \int_{t0}^{t1} \left\{ \sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext} + \sum_i \mathbf{Q} \right\} dt$



$t = t_0$ *Total Angular Momentum* $\mathbf{h}_0^{TOT}$

$t = t_1$ *Total Angular Momentum* $\mathbf{h}_1^{TOT}$

Example 8.7.1 In the 'ice skater trick' the skaters change their mass moment of inertia I_{Gzz} by retracting their arms. Find a formula for ω_1



Free spinning ⇒ no external moment acts in $\underline{k}$ direction on skater

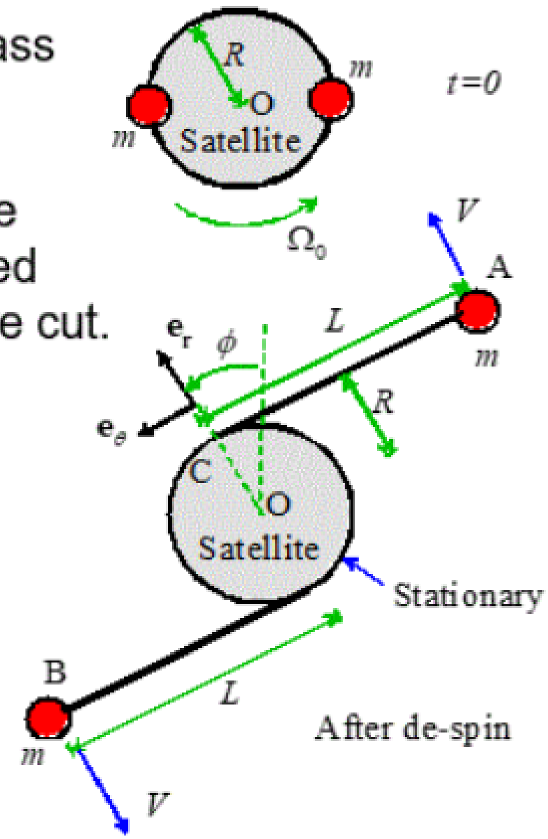
$$\Rightarrow A_z^{ext} = 0 \quad \Rightarrow \quad h_{z1} - h_{z0} = 0$$

$$h_{z0} = I_{Gzz0}\,\omega_0 \qquad h_{z1} = I_{Gzz1}\,\omega_1$$

$$\Rightarrow \omega_1 = \underbrace{\frac{I_{Gzz0}}{I_{Gzz1}}}\,\omega_0$$

$>1 \Rightarrow$ angular speed increases

Example 8.7.2 The figure shows a 'yoyo de-spin' of a satellite with mass moment of inertia I_{Gzz}. Two masses *m* are attached to the cylinder by inextensible tethers.
At time *t=0* the satellite spins with angular speed Ω_0 and the tethers are tightly wound around the cylinder. To de-spin, the masses are released and the tethers unwind. When the satellite comes to rest the tethers are cut. Find the length *L* of the tethers when they are cut.



No ext. forces $\Rightarrow \Delta W^{ext} = 0 \quad \underline{A}^{ext} = \underline{0}$

Hence $\quad T_1 + U_1 = T_0 + U_0$

$$\underline{h}_1^{TOT} = \underline{h}_0^{TOT}$$

Initial State $\quad U_0 = 0$

Circular motion $v = \omega R$

$$T_0 = \tfrac{1}{2} I_{Gzz} \Omega_0^2 + 2 \times \tfrac{1}{2} m \, (\Omega R)^2$$

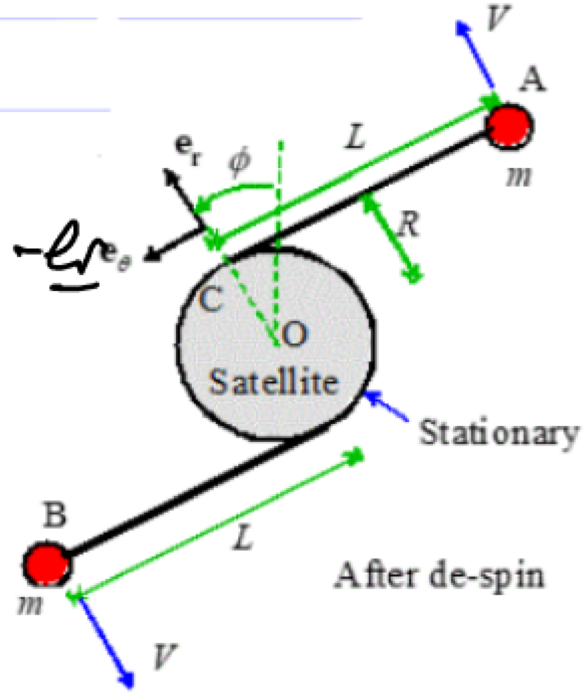
$$\Rightarrow T_0 = \tfrac{1}{2} (I_{Gzz} + 2mR^2) \Omega_0^2$$

$$\underline{h}_0 = I_{Gzz} \Omega_0 \underline{k} + 2R \, (m \Omega_0 R) \, \underline{k}$$

$$\Rightarrow \underline{h}_0 = (I_{Gzz} + 2mR^2) \Omega_0 \, \underline{k}$$

At release: Point C on tether is stationary

$$\underline{V}_A - \underline{V}_C = \frac{d\phi}{dt}\,\underline{k} \times (-L\,\underline{e}_\theta) \qquad \underline{k}\times\underline{e}_\theta = -\underline{e}_r$$



$$\Rightarrow \quad \underline{V}_A = L(d\phi/dt)\,\underline{e}_r$$

Hence $T_1 = 2\times\frac{1}{2}m\left(L\frac{d\phi}{dt}\right)^2$ (symmetry)

$$= mL^2(d\phi/dt)^2$$

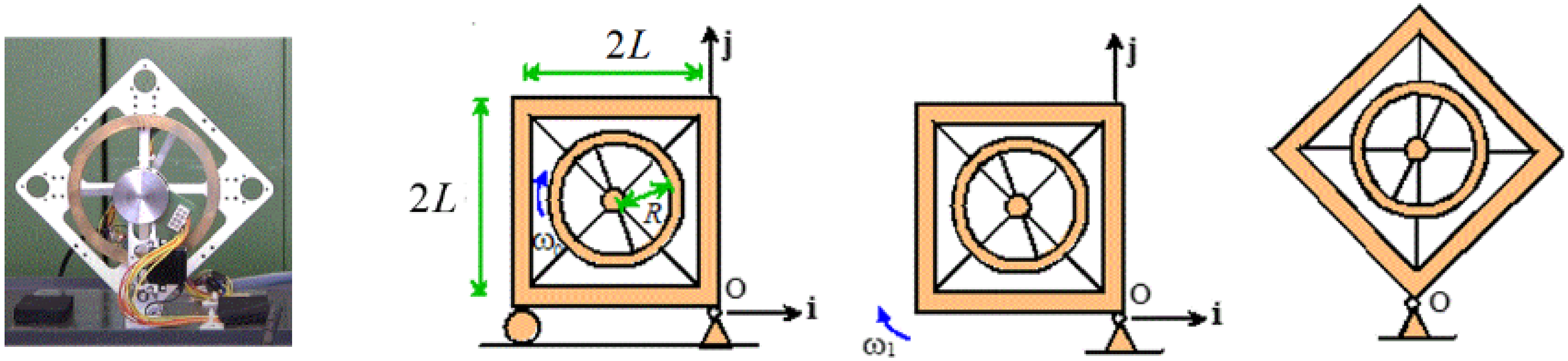
$$\underline{h}_1 = 2\,\underline{r}_A \times m\underline{V}_A = 2(-L\,\underline{e}_\theta + R\underline{e}_r)\times mL(d\phi/dt)\,\underline{e}_r$$

$$= 2mL^2(d\phi/dt)\,\underline{k}$$

$$T_1 = T_0 \Rightarrow \quad \tfrac{1}{2}(I_{Gzz} + 2mR^2)\Omega_0^2 = mL^2(d\phi/dt)^2 \qquad (1)$$

$$\underline{h}_1 = \underline{h}_0 \Rightarrow \quad (I_{Gzz} + 2mR^2)\Omega_0 = 2mL^2(d\phi/dt) \qquad (2)$$

$$\frac{(2)^2}{2\times(1)} \Rightarrow \quad 2mL^2 = I_{Gzz} + 2mR^2 \Rightarrow \boxed{L = R\sqrt{1 + \frac{I_{Gzz}}{2mR^2}}}$$



Example 8.7.3 The figure shows a 1-D prototype for the 'Cubeli.' The frame has total mass *M*, the ring has mass *m*.
At time *t=0* the frame is stationary and the ring spins with speed ω_0. The ring is then braked rapidly and locked to the frame. The frame and ring rotate together and come to rest with the frame balanced on a corner. Find the necessary speed ω_0

Preliminary calcs: Find I_{Gzz} of frame and I_{Ozz}^{total}

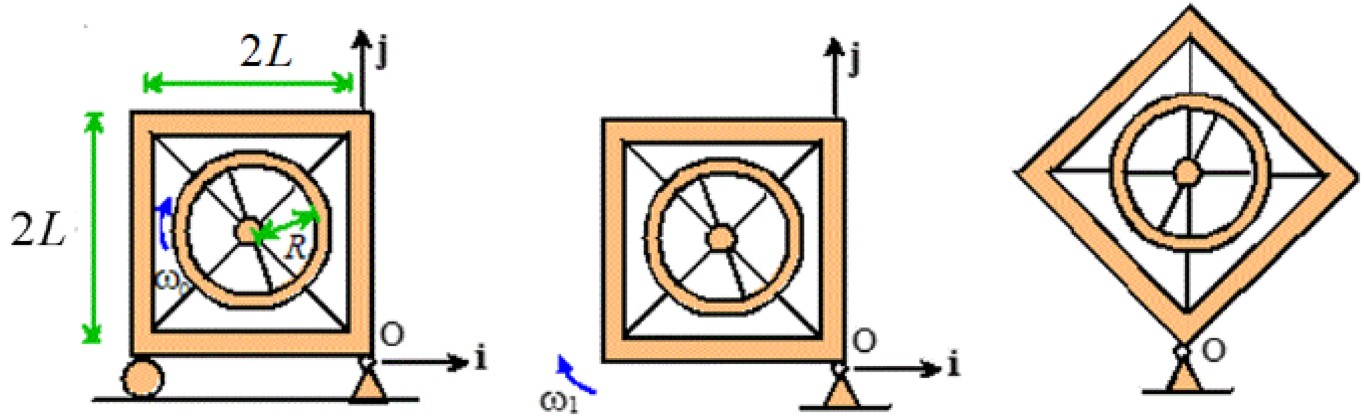
Frame: $$I_{Gzz} = 4\left\{\frac{M}{4}\frac{(2L)^2}{12} + \frac{M}{4}L^2\right\} = \frac{4}{3}ML^2$$

Total $$I_{Ozz} = \frac{4ML^2}{3} + mR^2 + (M+m)(\sqrt{2}L)^2$$

$$= \frac{10}{3}ML^2 + m(R^2 + 2L^2)$$

Energy conserved
for $t_1 < t < t_2$

$$T_2 + U_2 = T_1 + U_1$$



t_0 t_1 t_2

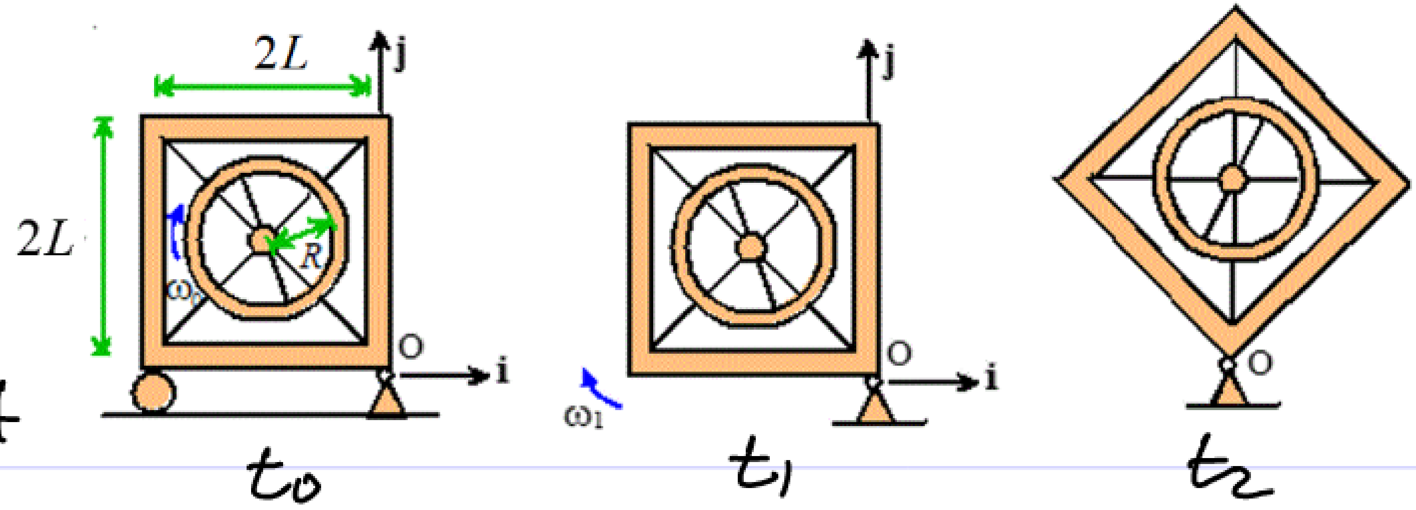
$$T_2 = 0 \qquad U_2 = (M+m)\, g \sqrt{2}\, L$$

$$T_1 = \frac{1}{2} I_{Ozz}\, \omega_f^2 \qquad U_1 = (M+m)\, g\, L$$

Hence $(M+m)\, g \sqrt{2}\, L = \frac{1}{2} I_{Ozz}\, \omega_f^2 + (M+m)\, g\, L$

$$\Rightarrow \boxed{\omega_f = \sqrt{\frac{2(M+m)\, g L (\sqrt{2}-1)}{I_{Ozz}}}}$$

When ring is braked reaction forces act at O



$\Rightarrow \underset{\sim}{r} \times \underset{\sim}{F} = \underset{\sim}{0}$ about O

Impulse of gravity is negligible $\Rightarrow \underset{\sim}{A}^{ext} = \underset{\sim}{0}$ about O

$\Rightarrow \underset{\sim}{h}_1 - \underset{\sim}{h}_0 = \underset{\sim}{0}$ about $\underset{\sim}{O}$

$\underset{\sim}{h}_1 = -I_{Ozz}\,\omega_f\,\underset{\sim}{k}$

$\underset{\sim}{h}_0 = \underset{\sim}{r}_G \times m\underset{\sim}{v}_G + I_{Gzz}\,\omega_z\,\underset{\sim}{k}$ for ring

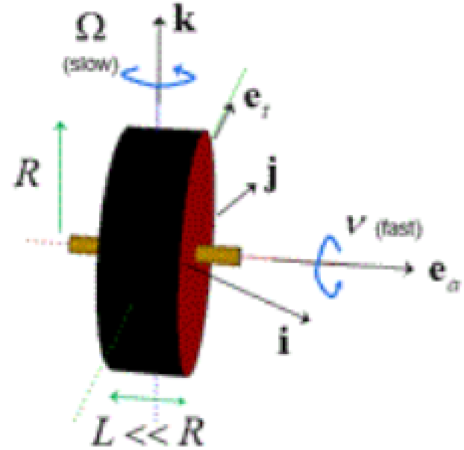
$= \underset{\sim}{0} + mR^2\omega_0(-\underset{\sim}{k})$

$\underset{\sim}{h}_1 = \underset{\sim}{h}_0 \Rightarrow \omega_0 = (I_{Ozz}/mR^2)\,\omega_f$

$$\Rightarrow \boxed{\omega_0 = \frac{\sqrt{2(M+m)gL(\sqrt{2}-1)}}{mR^2}\sqrt{\frac{10}{3}ML^2 + m(R^2+2L^2)}}$$

6.8 Examples of 3D motion of rigid bodies

Example 6.8.1 Find the pure moment required to cause steady precession of a spinning wheel



Assume COM stationary $\underline{v}_G = \underline{a}_G = \underline{0}$

Neglect gravity

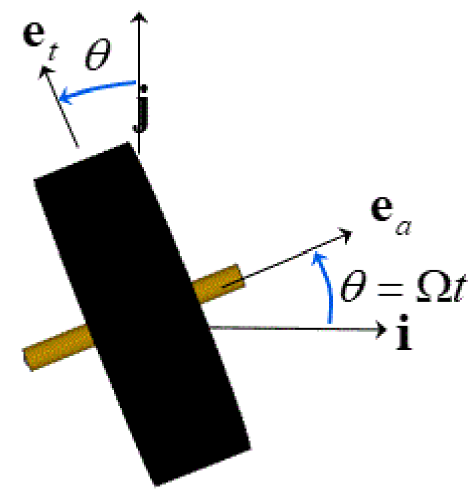
Angular Momentum $\underline{Q} = \dfrac{d\underline{h}}{dt}$

$$\underline{h} = I_G \underline{\omega} \qquad \underline{\omega} = \nu \underline{e}_a + \Omega \underline{k}$$

Assume I_G has form $\dfrac{I_a}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ eg $I_a = \frac{1}{2} m R^2$ for solid disk

$$\Rightarrow I_G \underline{\omega} = \frac{I_a}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \nu \\ 0 \\ \Omega \end{bmatrix} = \frac{I_a}{2}\begin{bmatrix} 2\nu \\ 0 \\ \Omega \end{bmatrix} \Rightarrow \underline{h} = I_a \nu \underline{e}_a + \frac{I_a}{2}\Omega \underline{k}$$

Note $\frac{d\underline{h}}{dt} \neq 0$ since $\underline{e}_a$ rotates $\Rightarrow \frac{d\underline{e}_a}{dt} \neq 0$

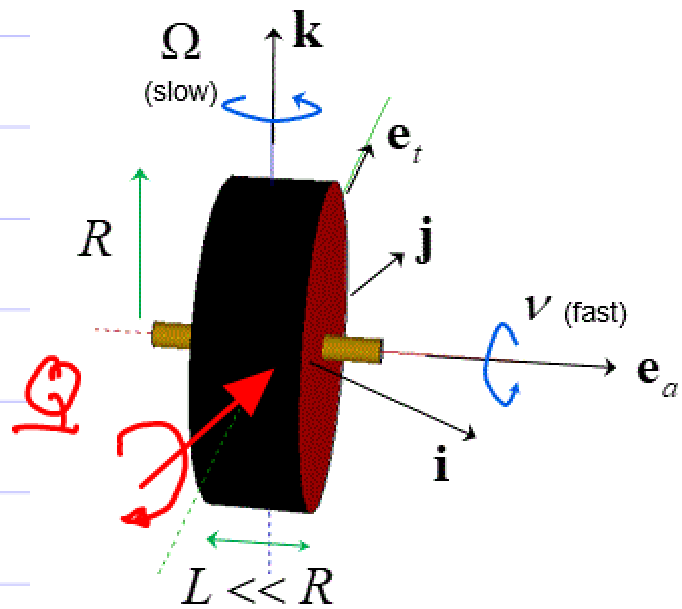


$$\underline{e}_a = \cos\theta\,\underline{i} + \sin\theta\,\underline{j}$$
$$\underline{e}_t = -\sin\theta\,\underline{i} + \cos\theta\,\underline{j}$$

$$\Rightarrow \frac{d\underline{e}_a}{dt} = \frac{d\theta}{dt}\left\{-\sin\theta\,\underline{i} + \cos\theta\,\underline{j}\right\} = \Omega\,\underline{e}_t$$

Hence $\boxed{\underline{Q} = \frac{d\underline{h}}{dt} = I_a \nu \Omega\, \underline{e}_t}$

Note $\underline{Q}$ is perpendicular to precession axis!

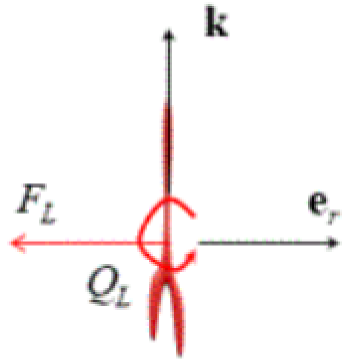


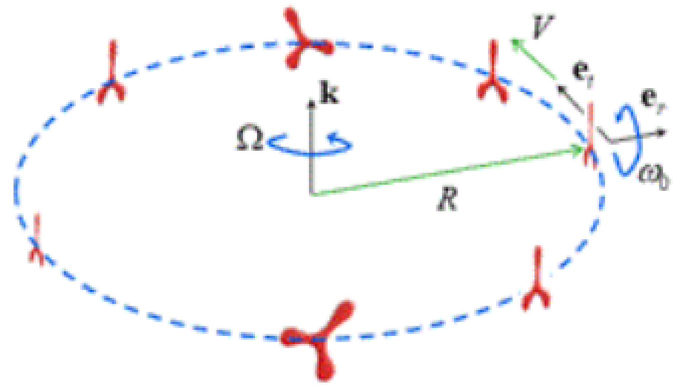
Precession rate $\boxed{\Omega = \frac{|\underline{Q}|}{I_a \nu}}$

Example 6.8.2 A boomerang is subjected to an aerodynamic lift force and moment

$$F_L = \frac{\pi}{4}\rho C_L L^2\left(V^2 + L^2\omega_0^2\right) \qquad Q_L = \frac{\pi}{4}\rho C_L L^4 V\omega_0$$

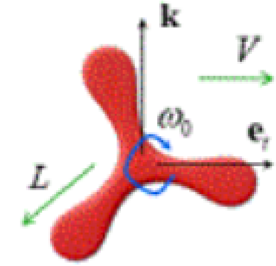
Find a formula for the radius of the path





Circular motion ⇒ precession rate $\Omega = \dfrac{V}{R}$

$$\mathbf{I}_G = \frac{mL^2}{6}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



From ex 6.8.1

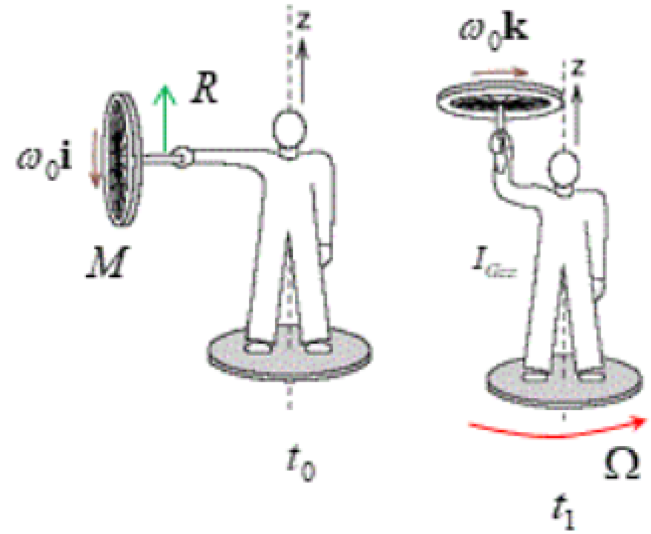
$$Q_L = I_a \omega_0 \Omega \qquad I_a = \frac{mL^2}{3}$$

Hence $$\frac{\pi}{4}\rho C_L L^4 \cancel{V} \cancel{\omega_0} = \frac{mL^2}{3}\cancel{\omega_0}\frac{\cancel{V}}{R}$$

$$\Rightarrow \boxed{R = \frac{4}{3\pi}\frac{m}{\rho C_L L^2}} \qquad \text{(Independent of throw)}$$

(Can also use F_L to show $\omega_0 = \sqrt{2}\, V/L$)

6.8.3 Example Spinning wheel on a freely rotating platform.
Wheel has (absolute) angular velocity shown.
Find the angular velocity of the platform at time t_1



Platform rotates freely

$\Rightarrow$ no external moment parallel to $\underline{k}$

$$\Rightarrow A_z^{ext} = 0 \qquad \text{Hence} \quad h_{z1} = h_{z0}$$

$$\underline{h}_0 = mR^2 \omega_0 \underline{i} \qquad \underline{h}_1 = mR^2 \omega_0 \underline{k} + I_{Gzz} \Omega \underline{k}$$

$$\text{Hence} \quad mR^2 \omega_0 + I_{Gzz} \Omega = 0$$

$$\Rightarrow \boxed{\Omega = -\frac{mR^2}{I_{Gzz}} \omega_0}$$